3. GOL'DSHTEIN R.V. and ENTOV V.M., Variational estimates for stress intensity factors on the contour of a plane crack of a normal discontinuity, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.3, 1979.
4. VLADIMIROV V.S., Generalized functions in mathematical physics, Nauka, Moscow, 1976.
5. RICE J., Mathematical methods in fracture mechanics /Russian translation/, Fracture, Vol. 2, Mir, Moscow, 1975.
6. MAZ'YA V.G., NAZAROV S.A., PLAMENEVSKII B.A., Asymptotic form of the solutions of elliptical boundary value problems for singular domain perturbations. Izdat. Tbilis. Univ., Tbilisi, 1981.
7. KALKER J.J., The surface displacement of an elastic half-space loaded in a slender bounded, curved surface region with application to the calculation of the contact pressure under the roller, J. Inst. Math. Applic., Vol.19, No.2, 1977.
8. JOHNSON R.E., An improved slender-body theory for Stokes Flow, J. Fluid Mech. Vol. 99, Pt. 2, 1980.
9. COLE J., Perturbation methods in applied mathematics /Russian translation/, Mir, Moscow, 1972.
10. NAYFEH A.H., Perturbation methods /Russian translation/, Mir, Moscow, 1976.
11. SMETANIN B.I., Problem of tension on an elastic space containing a plane annular slot, PMM, Vol.32, No.3, 1968.
12. MASTROJANNIS E.N. and KERMANIDIS T.B., An approximate solution of the annular crack problem, Intern. J. Numer. Meth. Engng., Vol.17, No.11, 1981.

Translated by M.D.F.

PMM U.S.S.R., Vol.48, No.5, pp.626-630, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

STATIONARY MOTIONS OF A GYROSTAT WITH AN ELASTIC ANNULAR PLATE AND THEIR STABILITY*

M.K. NABIULLIN

Using Rumyantsev methods /1-3/ in the Kuz'min form /4/, stationary motions are deduced for a gyrostat with a circular annular plate clamped by the inner contour in a housing, and sufficient conditions are obtained for their stability. The paper touches on a cycle of papers devoted to investigating the stability of systems with distributed parameters: elastic rods, flexible rectangular plates, and a flexible string /5-19/.

1. We introduce the following coordinate system: $Cx_1x_2x_3$ is the orbital system with origin at the centre of mass of the mechanical system for the plate state of strain, the Cx_2 axis is along the orbit radius, the Cx_3 axis is perpendicular to the orbit plane, and the axis Cx_1 is orthogonal to the Cx_2, Cx_3 axes; $Oxyz$ is the coordinate system coupled rigidly to the gyrostat housing whose axes are directed along the principal central axes constructed for the centre of mass O of the system for the undeformed state of the plate; $Cy_1y_2y_3$ is the coordinate system whose y_s axes $(s=1,2,3)$ are parallel to the x, y, z axes, respectively.

We will define the gyrostat location in the orbital coordinate system by the Euler angles ψ, θ, φ and the direction of the x_s axes $(s=1,2,3)$ with respect to the axes of the system $Cy_1y_2y_3$ by the direction cosines $\alpha_{s1}, \alpha_{s2}, \alpha_{s3}$ that depend in a known manner on the angles ψ, θ, φ, for instance, $\alpha_{31}=\sin\varphi\sin\theta$ [20].

We will define the location of points of the plate in the deformed state with respect to the gyrostat housing by a radius-vector whose projections on the axes are

$$r_x=(a+r)\cos\lambda-zu_1,\quad r_y=(a+r)\sin\lambda-zu_2 \tag{1.1}$$
$$r_z=z+w\quad (u_1=w_r\cos\lambda-(a+r)^{-1}w_\lambda\sin\lambda,\quad u_2=w_r\sin\lambda+(a+r)^{-1}w_\lambda\cos\lambda)$$

Here a is the radius of the inner circular contour of the middle plane located in the Oxy plane, $a+r, \lambda, z$ are cylindrical coordinates of an arbitrary point of the plate in the undeformed state, $w(r,\lambda,t)$ is the projection of the elastic displacement vector of an arbitrary point of the middle plane on the z-axis, and the letter subscripts on the quantity w denote first-order partial derivatives with respect to the variable indicated in the subscript.

*Prikl.Matem.Mekhan., Vol.48, 5, 864-867, 1984

The differential equations of motion and the boundary conditions of a gyrostat with an annular elastic plate in the restricted problem in a circular orbit allow of a Jacobi integral /21/ when the gyrostatic moments k_s $(s = 1, 2, 3)$ are constant.

$$H = T_1 + \frac{1}{2}\omega_0^2 \sum_{i,j=1}^{3} A_{ij}(3\alpha_{2i}\alpha_{2j} - \alpha_{3i}\alpha_{3j}) - \tag{1.2}$$

$$\frac{1}{2}\omega_0^2 \sum_{i=1}^{3} A_{ii} - \omega_0 \sum_{s=1}^{3} k_s\alpha_{3s} + \Pi = \text{const}$$

$$\Pi = \frac{D}{2}\int_{\tau_1} (a+r)\Big\{(\nabla^2 w)^2 +$$

$$2(1-\sigma)\Big[\Big(\frac{\partial}{\partial r}\frac{w_\lambda}{a+r} - w_{rr}\Big(\frac{w_{\lambda\lambda}}{(a+r)^2} + \frac{w_r}{a+r}\Big)\Big]\Big\} d\tau_1$$

$$\Big(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{(a+r)^2}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{a+r}\frac{\partial}{\partial r}, \int_{\tau_1} F\, d\tau_1 = \int_0^b\int_0^{2\pi} F\, dr\, d\lambda\Big)$$

where T_1 is the kinetic energy in relative motion, Π is the potential energy of the plate, σ is Poisson's ratio, ω_0 is the orbital angular velocity, and $D/2$ is the plate cylindrical stiffness; the double letter subscript for the quantity w in the expression for the plate potential energy denotes the second partial derivative with respect to the coordinates indicated in the subscript, and A_{ij} $(i, j = 1, 2, 3)$ are tensor components of the system inertia constructed for the centre of mass of the system C.

2. The equations and boundary conditions obtained by equating the first variation of the integral (1.2) to zero allow of solutions corresponding to the equilibrium locations in the orbital coordinate system (the vector projections relative to the angular velocity ω_s on the y_s axes $(s = 1, 2, 3)$ are zero).

Three families of relative equilibrium locations exist when the circular annular plate is not deformed $(w = w_r = w_\lambda = 0)$ and its middle plane is either orthogonal to the orbit radius $(\psi_0 = 0,\ \theta_0 = \pi/2,\ \varphi = \varphi_0)$ or tangent to the trajectory of the centre of mass $(\psi_0 = \theta_0 = \pi/2,\ \varphi = \varphi_0)$, or coincides with the orbit plane. The principal central moments of inertia of the sytem $A_s (s = 1, 2, 3)$ relative to the x, y, z axes for the undeformed state of the plate and the gyrostatic moments k_s $(s = 1, 2, 3)$, and the angle φ_0 should satisfy the following relationships for the first family of motions:

$$(A_1 - A_2)\,\omega_0^2 \sin\varphi_0\cos\varphi_0 + \omega_0(k_1\cos\varphi_0 - k_2\sin\varphi_0) = 0,\quad k_3 = 0$$

For the second family of motions the coefficient $A_1 - A_2$ is replaced by $4(A_1 - A_2)$.

3. To investigate the stability of the stationary motions obtained by using Rumyantsev's theorem /2/, we take the Liapunov functional in the form $V = H - H_0$, where H_0 is the value of the integral (1.2) evaluated along the unperturbed motions, and we consider the sign-definiteness condition of its second variation $\delta^2 H$, which equals the sum of the second variations of the kinetic energy $\delta^2 T_1$ in relative motion and the potential energy $\delta^2\Pi_1$ of the system.

Establishment of the sign-definiteness of the Liapunov functional is given a foundation below by the idea of introducing the integral characteristics of the motion of continuous media proposed by Rumyantsev when investigating the motion-stability of complex systems relative to part of the variables /1/.

It can be shown that the second variation of the kinetic energy $\delta^2 T_1$ is positive-definite and continuous in the metric

$$P_1 = \sum_{s=1}^{3}\omega_s^2 + \int_{\tau_1}\rho_1(a+r)\Big[v_{11}^2 + \frac{h^2}{3}(v_{22}^2 + v_{33}^2)\Big] d\tau_1 + z_c^{\cdot 2}$$

$$v_{11} = w^\cdot - z_c^\cdot + (a+r)(\omega_1\sin\lambda - \omega_2\cos\lambda),\quad v_{22} = w_r^\cdot + \omega_1\sin\lambda - \omega_2\cos\lambda$$

$$v_{33} = w_\lambda^\cdot (a+r)^{-1} + \omega_1\cos\lambda + \omega_2\sin\lambda,\quad \rho_1 = 2h\rho$$

Here $2h$ is the plate thickness, ρ is its density, and z_c is the coordinate of the centre of mass C of a system in the $Oxyz$ coordinate system. We retain the previous notation for the deviations of the variables from their unperturbed values and find the minimum μ of the functional

$$\Phi = \frac{2\Pi}{(f_2 D)},\quad f_2 = \int_{\tau_1}(a+r)\Big[w^2 + \frac{h^2}{3}\Big(w_r^2 + \frac{w_\lambda^2}{(a+r)^2}\Big)\Big] d\tau_1 \tag{3.1}$$

in the class of functions D_4 that have continuous partial derivatives in the domain $\tau_1 = \{r, \lambda: 0 \leqslant r \leqslant b, 0 \leqslant \lambda \leqslant 2\pi\}$ in the variables r, λ to the fourth order inclusive, and satisfy the boundary conditions

$$r = 0, \quad t \geqslant t_0, \quad w = w_r = 0 \tag{3.2}$$

Equating the first variation of the functional (3.1) to zero, we obtain the following boundary value problem:

$$\nabla^4 w + \alpha^2 \nabla^2 w - \mu w = 0 \qquad (\alpha^2 = {}^1/_3 \mu h^2) \tag{3.3}$$

$$r = b, \quad t \geqslant t_0, \quad w_{rr} + \sigma w_{\lambda\lambda} n_5^2 + \sigma w n_5 = 0 \tag{3.4}$$

$$w_{rrr} + (2 - \sigma) w_{\lambda\lambda r} n_5^2 - (\sigma + 3) w_{\lambda\lambda} n_5^3 + \frac{1}{3} \mu h^2 w_r + w_{rr} n_5 - w_r n_5^2 = 0 \qquad (n_5 = (a + r)^{-1})$$

It can be shown that the expression

$$w = [c_1 J_m (\beta (a + r)) + c_2 Y_m (\beta (a + r)) + c_3 I_m (\gamma (a + r)) + c_4 K_m (\gamma (a + r))] \cos m\lambda \tag{3.5}$$

$$\beta = \left(\frac{1}{2}\alpha^2 + k\right)^{1/2}, \qquad \gamma = \left(-\frac{1}{2}\alpha^2 + k\right)^{1/2}, \qquad k = \left(\mu + \frac{1}{36}\mu^2 h^4\right)^{1/2}$$

is the solution of (3.3), where J_m, Y_m, I_m, K_m are Bessel functions (of the first and second kinds, and modified, respectively) with orders $m = 0, 1, 2, \ldots$

Substitution of the solution (3.5) into the boundary conditions (3.2) and (3.4) results in a system of algebraic equations in the arbitrary constants c_i $(i = 1, 2, 3, 4)$ with coefficients dependent on the Bessel functions. Equating the determinant of this system to zero, we obtain a transcendental frequency equation to find the desired minimum μ of the functional (3.1). The frequency equation is not presented because of its awkwardness.

From (3.1) we find an estimate of the form

$$Z^2 = 2\Pi - \rho_1 \varkappa f_2 \geqslant 0 \qquad (\tau = \mu D/\rho_1) \tag{3.6}$$

We now introduce new variables and functional-integral characteristics by the formulas

$$x_1 = (a + r)^{1/2} w \sin(\lambda + \varphi_0), \quad x_2 = (a + r)^{1/2} w \cos(\lambda + \varphi_0) \tag{3.7}$$

$$x_3 = (a + r)^{1/2}(u_2 \cos\varphi_0 + u_1 \sin\varphi_0), \qquad x_4 = (a + r)^{1/2}(u_2 \sin\varphi_0 - u_1 \cos\varphi_0), \qquad y_i = \int_{\tau_1} \rho_1 (a + r)^{3/2} x_i \, d\tau_1 \qquad (i = 1, 2)$$

$$y_j = \int_{\tau_1} \rho_1 (a + r)^{1/2} x_j \, d\tau_1 \qquad (j = 3, 4)$$

The dependence

$$x_1^2 + x_2^2 = (a + r) w^2, \quad x_3^2 + x_4^2 = (a + r)(w_r^2 + n_5^2 w_\lambda^2) \tag{3.8}$$

evidently holds between the initial variables w, w_r, w_λ and the new variables x_i $(i = 1, 2, 3, 4)$.

We apply the Cauchy inequality to the functionals y_i $(i = 1, 2, 3, 4)$; we then obtain

$$z_i^2 = C \int_{\tau^1} \rho_1 x_i^2 \, d\tau_1 - y_i^2 \geqslant 0 \qquad (i = 1, 2) \tag{3.9}$$

$$z_j^2 = B \int_{\tau_1} \rho_1 x_j^2 \, d\tau_1 - y_j^2 \geqslant 0 \qquad (j = 3, 4)$$

Here C is the moment of inertia of the plate relative to the z-axis, and B is its mass.

By using the relationships (3.6)-(3.9), an expression can be written for the second variation of the potential energy $\delta^2\Pi_1$ in these variables. The conditions for its sign-definiteness result in the inequalities

$$\varkappa - 3\omega_0^2 > 0, \qquad A_2\omega_0^2 + \frac{k_2\omega_0}{\cos\varphi_0} > \max(l_{11}, l_{22}) \tag{3.10}$$

$$\Delta_5^{(1)} = 3\omega_0^2\left(A_1 \cos^2\varphi_0 + A_2 \sin^2\varphi_0 - A_3 - Bh^2 \frac{\omega_0^2}{\varkappa} - 3C \frac{\omega_0^2}{\varkappa - 3\omega_0^2}\right) > 0$$

$$l_{11} = \omega_0^2 (A_1 \cos^2\varphi_0 + A_2 \sin^2\varphi_0), \qquad l_{22} = 4A_3\omega_0^2 - 3\omega_0^2\left(A_1 \sin^2\varphi_0 + A_2 \cos^2\varphi_0\right) + 16Bh^2 \frac{\omega_0^4}{3(\varkappa + \omega_0^2)} + 16C \frac{\omega_0^4}{\varkappa - 3\omega_0^2} + \frac{9\omega_0^4}{\Delta_5^{(1)}} (A_2 - A_1)^2 \sin^2\varphi_0 \cos^2\varphi_0$$

When these inequalities are satisfied, the functional $\delta^2\Pi_1$ is positive-definite and continuous in the metrics

$$P_2 = \psi^2 + \theta^2 + \varphi^2 + C \int_{\tau_1} \rho_1 (a + r) w^2 d\tau_1 + B \int_{\tau_1} \rho_1 (a + r) ({w_r}^2 + n_5 {w_\lambda}^2) d\tau_1 + {z_c}^2, \qquad P_3 = P_2 + \int_{\tau_1} (a + r) \left[{w_{rr}}^2 + (w_r n_5 + w_{\lambda\lambda} {n_5}^2)^2 + \left(\frac{\partial}{\partial r} n_5 w_\lambda \right)^2 \right] d\tau_1$$

According to Rumyantsev's theorem /2/, inequality (3.10) is the sufficient condition for stability of the first family of equilibrium positions in the metrics $P_1 + P_2$ and $P_1 + P_3$. When inequalities (3.10) are satisfied, the Liapunov functional also satisfies the conditions of the theorem /22/.

It follows from the inequalities obtained that the sufficient conditions for stability depend substantially on the lowest natural vibration frequency for the circular annular plate and its parameters; a diminution in the plate cylindrical stiffness can result in destabilization of the family of equilibrium positions.

The inequalities (3.10) generalize the sufficient conditions for stability of a satellite-gyrostat without deformable elements and reduce to the criteria in /23/ as $\varkappa \to \infty$.

The author is grateful to V.M. Matrosov for his interest and for useful discussions.

REFERENCES

1. MOISEEV N.N. and RUMYANTSEV V.V., Dynamics of a body with fluid-filled cavities, Nauka, Moscow, 1965.
2. RUMYANTSEV V.V., On the motion and stability of an elastic body with a fluid-filled cavity, PMM, Vol.33, No.6, 1969.
3. RUMYANTSEV V.V., Certain problems of the dynamics of complex systems. Problems of applied mathematics and mechanics, Nauka, Moscow, 1971.
4. KUZ'MIN P.A., Stationary motions of a solid and their stability in a central gravitational field, Trans. on Interuniversity Conference on Appl. Theories of Motion Stability and Analytical Mechanics, Kazan', 1964.
5. MOROZOV V.M., RUBANOVSKII V.N., RUMYANTSEV V.V. and SAMSONOV V.A., On the bifurcation and and stability of steady motions of complex mechanical systems. PMM, Vol.37, No.3. 1973.
6. RUBANOVSKII V.N. On the stability of certain motions of a solid with elastic rods and a fluid, PMM, Vol.36, No.1, 1972.
7. RUBANOVSKII V.N., Stability of relative equilibrium in the circular orbit of a solid with elastic rods performing bending-torsional vibrations, Theoretical and Applied Mechanics, Vol.3, No.2, 1972.
8. MOROZOV V.M. and RUBANOVSKII V.N., Stability of the relative equilibrium of a solid with two elastic rods, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1974.
9. RUBANOVSKII V.N., Stability of stationary rotations of a heavy solid with two elastic rods, PMM, Vol.40, No.1, 1976.
10. RUBANOVSKII V.N., Stability of stationary rotations of a free solid with a fluid-filled elastic cylindrical shell during system motion by inertia. Stability of Motion. Analytical Mechanics. Motion Control, Nauka, Moscow, 1981.
11. MEIROVITCH L., Stability of a spinning body containing elastic parts via Liapunov's direct method, AIAA Journal, Vol.8, No.7, 1970.
12. MEIROVITCH L., A method for the Liapunov stability analysis of force-free dynamical systems. AIAA Journal, Vol.9, No.9, 1971.
13. MEIROVITCH L. and CALICO R.A., A comparative study of stability methods for flexible satellites, AIAA Journal, Vol.11, No.1, 1973.
14. MEIROVITCH L., Liapunov stability analysis of hybrid dynamical systems in the neighbourhood of non-trivial equilibrium, AIAA Journal, Vol 12, No.7, 1974.
15. PASCAL M., La stabilite d'attitude d'un satellite muni de panneaux solaires, Acta Astronaut., Vol.5, No.10, 1978.
16. NABIULLIN M.K., On stability of stationary motion of a gyrostat with elastic plates in a Newtonian central force field, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1981.
17. BOLOTINA N.E., VIL'KE V.G., On the stability of the equilibrium positions of a flexible heavy filament attached to a satellite in a circular orbit, Kosmich. Issled., Vol.16, No.4, 1978.
18. KUZ'MIN P.A., On the stability ofthe circular shape of a flexible filament, Trudy, Kazansk. Aviats. Inst., No.20, 1948.
19. KUZ'MIN P.A., Stability of the circular filament shape with a countable set of degrees of freedom, Trudy Kazansk. Aviats. Inst., No.22, 1949.
20. LUR'E A.I., Analytical Mechanics, Fizmatgiz, Moscow, 1961.
21. BELETSKII V.V., Motion of an Artificial Satellite Relative to the Centre of Mass, Nauka, Moscow, 1965.
22. MOVCHAN A.A., Stability of processes in two metrics, PMM Vol.24, No.6, 1960.

23. RUMYANTSEV V.V., On the stability of stationary satellite motion, Vyschitel. Tsentr Akad. Nauk SSSR, 1967.

Translated by M.D.F.

PMM U.S.S.R.,Vol.48,No.5,pp. 630-634,1984
Printed in Great Britain.

0021-8928/84 $10.00+0.00

ON AN INTEGRAL EQUATION OF CONTACT PROBLEMS OF ELASTICITY THEORY IN THE PRESENCE OF ABRASIVE WEAR*

E.V. KOVALENKO

An algorithm based on the method of matched asymptotic expansions and enabling one to avoid mathematical incorrectness is proposed for solving the integral equations of contact problems taking abrasive wear of the surfaces of contiguous bodies into account. An exact solution is written for the convolution type integral equation of the second kind with a logarithmic kernel in a semi-infinite interval in the class of continuous functions that vanish at infinity.

A mathematical inaccuracy is committed in solving the integral equations of contact problems of elasticity theory in the presence of abrasive wear (/1-4/, etc.). The quantity characterizing the contact pressure distribution law and have a singularity of the square-root type for $t = 0$ at the ends of the contact domain /5/ was expanded in a Fourier series in the eigenfunctions of a certain self-adjoint completely continuous integral operator acting in a space of square-summable functions. However, as follows from the general theory of Fourier series in Hilbert spaces /6/, such a series will be known to be divergent in the norm of the space $L_2(-1, 1)$.

The approach proposed below enables one to avoid this mathematical incorrectness and in conjunction with the method in /7,8/ enable a solution of the contact problems mentioned to be constructed in the whole range of time variation. The closed solution of the convolution type integral equation of the second kind with logarithmic kernel in a semi-infinite interval can also be used to investigate contact problems for rough elastic bodies (or to study contact problems in the presence of thin elastic coatings) /9/ when the coefficient of the main term of the integral equation tends to zero.

1. The initial equations of the contact problem of elasticity theory for a linearly deformable base of general type in the presence of abrasive nwear can be written in the form /4/

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi,t)\,k\left(\frac{\xi-x}{\lambda}\right)d\xi = \gamma(t) - f(x) - \int_{0}^{t}\varphi(x,\tau)\,V(\tau)\,d\tau \tag{1.1}$$

$$(|x| \leqslant 1,\ 0 \leqslant t \leqslant T < \infty)$$

$$P(t) = \int_{-1}^{1}\varphi(x,t)\,dx \tag{1.2}$$

The piecewise-smooth function $V(t) \geqslant 0\ (0 \leqslant t \leqslant T)$ and the kernel $k(z)$ of the integral equation (1.1) is representable in the form

$$k(z) = \int_{0}^{\infty} L(u)\cos(uz)\,du, \quad z = \frac{\xi - x}{\lambda} \tag{1.3}$$

$$L(u) > 0,\ (|u| < \infty),\ L(u) = A + O(u^2)\ (u \to 0,\ A = \text{const})$$

$$L(u) = u^{-1} + O(u^{-3})\ (u \to \infty)$$

The analysis presented below refers to the case of an even function $f(x)$. The general case is considered analogously.

On the basis of (1.3), the following lemma is proved /5/:

Lemma. For all values of $0 \leqslant |z| < \infty$ the following representation holds for $k(z)$

---

*Prikl.Matem.Mekhan.,Vol.48,5 868-873,1984